A *p*-adic analytic approach to the absolute Grothendieck conjecture

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RIMS

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- \overline{K} : an algebraic closure of K
- U: a geometrically connected scheme of finite type over \boldsymbol{K}
- $U_{\overline{K}} := U \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$
- $\xi:$ a geometric point of U
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The Galois correspondence

There is a 1-1 correspondence:

$$\mathcal{H} \subset \pi_1(U)$$
: an open subgroup $igcup$ 1-1
 $U_{\mathcal{H}}$: a connected finite étale covering of U

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The homotopy exact sequence

We have the following exact sequence:

$$1 \to \pi_1(U_{\overline{K}}) \to \pi_1(U) \to G_K \to 1.$$

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Definition of hyperbolic curves

$$U$$
: a hyperbolic curve (over K) $\stackrel{\text{def}}{\iff} 2g + n - 2 > 0$.

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The Galois-theoretic interpretation of (*)

The following two exact sequences are canonically identified:

$\widetilde{X}:$ the integral closure of X in $\widetilde{K_U}$

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Definition of decomposition groups

For each closed point $\widetilde{x} \in \widetilde{X}$,

$$D_{\widetilde{x}} := \{ \gamma \in \pi_1(U) \, | \, \gamma(\widetilde{x}) = \widetilde{x} \}.$$

We refer to $D_{\widetilde{x}}$ as the *decomposition group* of \widetilde{x} .

The relative Grothendieck conjecture

Is it possible to recover U group-theoretically from $\pi_1(U)\twoheadrightarrow G_K?$ i.e.,

$$(\pi_1(U) \twoheadrightarrow G_K) \xrightarrow{?}_{\mathsf{recoverable}} U.$$

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Known affirmative results

- K/\mathbb{Q} : finitely generated, g = 0 [Nakamura, 1990]
- K/\mathbb{Q} : finitely generated, $n \neq 0$ [Tamagawa, 1997]
- K: sub-p-adic

(i.e. $K \simeq$ a subfield of a finitely generated extension of \mathbb{Q}_p) [Mochizuki, 1999]

The absolute Grothendieck conjecture

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Is it possible to recover U group-theoretically, solely from $\pi_1(U)$ (not $\pi_1(U) \twoheadrightarrow G_K$)? i.e., $\pi_1(U) \stackrel{?}{\rightsquigarrow} U.$

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Known affirmative results

- $[K:\mathbb{Q}]<\infty$ [Mochizuki, 2004]
- $p \ge 5$, K/\mathbb{Q}_p : unramified and finite, U: a "canonical lifting" [Mochizuki, 2003]
- $[K:\mathbb{Q}_p]<\infty$, U: "of Belyi type" [Mochizuki, 2007]

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However, when $[K : \mathbb{Q}_p] < \infty$, the absolute Grothendieck conjecture is **unsolved in general**.

For i = 1, 2, K_i : an algebraic number field G_{K_i} : the absolute Galois group of K_i Then

 $G_{K_1} \simeq G_{K_2} \Longleftrightarrow K_1 \simeq K_2.$

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However, when K is a p-adic local field, the analogue of the theorem of Neukirch-Uchida fails to hold.

In the following, we concentrate on the **absolute** p-adic Grothendieck conjecture.

Notation

In the following,

 K/\mathbb{Q}_p : a finite extension k: the residue field of K $q = q(K) := \sharp k$

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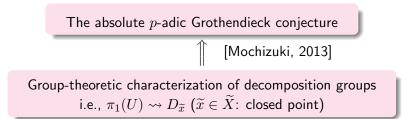
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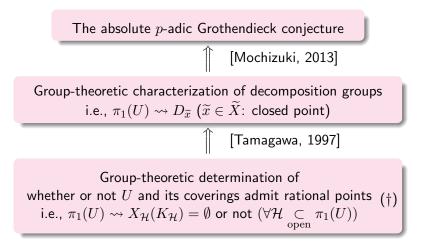
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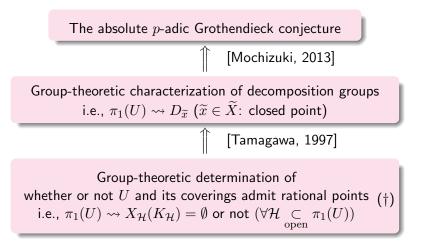
Moreover,

For each open subgroup $\mathcal{H} \subset \pi_1(U)$, $U_{\mathcal{H}}$: the étale covering of U corresponding to \mathcal{H} $X_{\mathcal{H}} := (U_{\mathcal{H}})^{\text{cpt}}$: the smooth compactification of $U_{\mathcal{H}}$ $K_{\mathcal{H}}$: the integral closure of K in $U_{\mathcal{H}}$ $q_{\mathcal{H}} := q(K_{\mathcal{H}})$

The absolute p-adic Grothendieck conjecture







In the following, we concentrate on (\dagger) .

Theorem (Serre)

Y: a nonempty and compact analytic manifold over KThen Y is the disjoint union of a finite number of (closed) balls and the number of balls is well defined $\mod (q-1)$.

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Definition of *i*-invariants

For Y as above,

$$i_K(Y) := (\text{the "number of balls"}) \in \mathbb{Z}/(q-1)\mathbb{Z}.$$

We refer to $i_K(Y)$ as the *i-invariant* of Y over K. Moreover, we set

$$i_K(\emptyset) \equiv 0 \mod (q-1).$$

 \mathcal{O}_K : the ring of integers of K \mathfrak{M}_K : the maximal ideal of \mathcal{O}_K \mathcal{O}_K : the ring of integers of K \mathfrak{M}_K : the maximal ideal of \mathcal{O}_K

Example

For $m \in \mathbb{Z}_{\geq 0}$,

$$i_K(\mathfrak{M}_K^m) \equiv 1 \mod (q-1),$$

where $\mathfrak{M}_{K}^{0} := \mathcal{O}_{K}$. Moreover,

$$i_K(\mathfrak{M}_K^m \setminus \mathfrak{M}_K^{m+1}) \equiv 0 \mod (q-1).$$

$$i_{K}(X(K)) \not\equiv 0 \mod (q-1)$$

$$\downarrow \qquad \bigstar$$

$$X(K) \neq \emptyset$$

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So, in some sense, the i-invariants are "weaker" data than the data of whether $X(K)=\emptyset$ or not.

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⇒ The group-theoretic recovery of the former data is **easier** than that of the latter?

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(A) May the decomposition groups be recovered from the data of the *i*-invariants of the sets of rational points of the hyperbolic curve and its coverings?

i.e., $i_{K_{\mathcal{H}}}(X_{\mathcal{H}}(K_{\mathcal{H}}))(\forall \mathcal{H} \underset{\text{open}}{\subset} \pi_1(U)) \xrightarrow{?} D_{\widetilde{x}} (\widetilde{x} \in \widetilde{X} : \text{ closed point})$

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(B) May the *i*-invariants of the set of rational points of the hyperbolic curve be recovered group-theoretically from the arithmetic fundamental group of the curve?
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(B) May the *i*-invariants of the set of rational points of the hyperbolic curve be recovered group-theoretically from the arithmetic fundamental group of the curve?
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Today, we give

- a complete affirmative answer to (A);
- a partial affirmative answer to (B).

An affirmative answer to (A)

The following theorem gives an affirmative answer to (A):

Theorem A (M)

Suppose:

- X is a proper hyperbolic curve over K;
- $q \neq 2$;
- $m \in \mathbb{Z}_{>1}$ is a divisor of q-1.

Then the following 5 conditions are equivalent:

(i)
$$X(K) \neq \emptyset$$
.

- (ii) $\exists X' \to X$: a finite étale covering s.t. $X'(K) \neq \emptyset$.
- (iii) $\exists X' \to X$: a finite étale covering s.t. $i_K(X'(K)) \not\equiv 0 \mod (q-1)$.
- (iv) $\exists X' \to X$: a finite étale covering s.t. $i_K(X'(K)) \not\equiv 0 \mod m$.
- (v) $\exists X' \to X$: a finite étale covering s.t. $i_K(X'(K)) \equiv$ (a power of p) mod (q-1).

Notation and assumption

For i = 1, 2, p_i : a prime K_i/\mathbb{Q}_{p_i} : a finite extension $q_i := q(K_i)$ U_i : a hyperbolic curve over K_i $X_i := U_i^{\text{cpt}}$: the smooth compactification of U_i

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Moreover, for each open subgroup $\mathcal{H}_i \subset \pi_1(U_i)$, we define

$$(U_i)_{\mathcal{H}_i}, (X_i)_{\mathcal{H}_i} (= (U_i)_{\mathcal{H}_i}^{\mathrm{cpt}}), (K_i)_{\mathcal{H}_i}, (q_i)_{\mathcal{H}_i} (:= q((K_i)_{\mathcal{H}_i}))$$

as above.

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as above.

Suppose that we are given an isomorphism $\alpha : \pi_1(U_1) \xrightarrow{\sim} \pi_1(U_2)$. Then we have $p_1 = p_2 =: p, q_1 = q_2 =: q$. [Mochizuki]

A group-theoretic consequence of Theorem A

The following theorem is a group-theoretic consequence of Theorem A:

Theorem A'(M)

Suppose that $\exists \mathcal{H}_0 \subset \pi_1(U_1)$: an open subgroup, $\exists m \in \mathbb{Z}_{>1}$: a divisor of $(q_1)_{\mathcal{H}_0} - 1$, $\forall \mathcal{H} \subset \pi_1(U_1)$: an open subgroup s.t. $\mathcal{H} \subset \mathcal{H}_0$,

$$i_{(K_1)_{\mathcal{H}}}((X_1)_{\mathcal{H}}((K_1)_{\mathcal{H}})) \equiv i_{(K_2)_{\alpha(\mathcal{H})}}((X_2)_{\alpha(\mathcal{H})}((K_2)_{\alpha(\mathcal{H})})) \mod m.$$
$$\cdots (\star)_{\mathcal{H}, r}$$

Then $\alpha : \pi_1(U_1) \xrightarrow{\sim} \pi_1(U_2)$ preserves decomposition groups. In particular, α arises from a unique isomorphism of schemes $U_1 \xrightarrow{\sim} U_2$.

A partial affirmative answer to (B)

The following theorem gives a partial affirmative answer to (B):

Theorem B (M)

Let $\mathcal{H} \subset \pi_1(U_1)$ be an open subgroup. Suppose:

- p is odd (in particular, 2 | (q-1));
- $g((X_1)_{\mathcal{H}}) \geq 2;$
- $(X_1)_{\mathcal{H}}$ has log smooth reduction

(i.e. has stable reduction after tame base extension).

Then $(\star)_{\mathcal{H},2}$ holds.

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Remark

If Theorem B is proved without assuming that $(X_1)_{\mathcal{H}}$ has log smooth reduction, the absolute *p*-adic Grothendieck conjecture holds for odd *p*.

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- $m \in \mathbb{Z}_{>1}$ is a divisor of q-1.

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The implications $(v) \Longrightarrow (iv) \Longrightarrow (ii) \Longrightarrow (ii) \Longrightarrow (i)$ are trivial. We will show the implication $(i) \Longrightarrow (v)$.

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 $\begin{array}{l} J: \mbox{ the Jacobian of } X \\ \mbox{ If } X(K) \neq \emptyset, \mbox{ for } P_0 \in X(K), \\ j = j_{P_0}: X \hookrightarrow J, \ P \mapsto [\mathscr{L}(P-P_0)]: \mbox{ a closed immersion} \end{array}$

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For $\nu \in \mathbb{Z}_{>0}$, we define an étale covering X_{ν} of X by:



where $\nu_J: J \to J$ denotes multiplication by ν on J.

Fact (M)

(i) For $n \gg 0$ and an appropriate choice of P_0 ,

$$i_K(X(K) \cap p_J^n(J(K))) \equiv (\text{a power of } p) \mod (q-1).$$
(Note that $X(K) \stackrel{j_{P_0}}{\hookrightarrow} J(K).$)

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 $i_K(X_{p^n}(K)) \equiv i_K(X(K) \cap p_J^n(J(K))) \times \sharp J(K)[p^n] \mod (q-1).$

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By Facts (i) and (ii), for $n \gg 0$ and an appropriate choice of P_0 ,

$$i_K(X_{p^n}(K)) \equiv (a \text{ power of } p) \mod (q-1).$$

We may take $X' = X_{p^n}$.

Theorem B (M)

Let $\mathcal{H} \subset \pi_1(U_1)$ be an open subgroup. Suppose:

- p is odd (in particular, $2 \mid (q-1)$);
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- (X₁)_H has log smooth reduction (i.e. has stable reduction after tame base extension).

Then $(\star)_{\mathcal{H},2}$ holds.

By replacing U_1 by $(U_1)_{\mathcal{H}}$, we may assume that $\mathcal{H} = \pi_1(U_1)$.

For i = 1, 2, by Deligne-Mumford, $\exists L_i/K_i$: a finite Galois extension, $X_i \times_{\text{Spec } K_i} \text{Spec } L_i$ has the stable model \mathfrak{X}_i .

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Note that α induces the following commutative diagram [Mochizuki]:

$$\pi_1(U_1) \longrightarrow \pi_1(X_1) \longrightarrow \operatorname{Gal}(\overline{K_1}/K_1)$$

$$\alpha_{\downarrow \wr} \qquad \alpha_{X}_{\downarrow \wr} \qquad \alpha_{K}_{\downarrow \wr}$$

$$\pi_1(U_2) \longrightarrow \pi_1(X_2) \longrightarrow \operatorname{Gal}(\overline{K_2}/K_2)$$

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$$\pi_{1}(U_{2}) \longrightarrow \pi_{1}(X_{2}) \longrightarrow \operatorname{Gal}(\overline{K_{2}}/K_{2})$$

By Fact (i), we may assume that $\alpha_K^{-1}(\operatorname{Gal}(\overline{K_2}/L_2)) = \operatorname{Gal}(\overline{K_1}/L_1)$.

Fact

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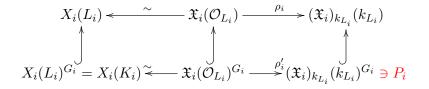
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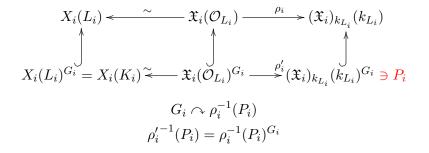
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By assumption and Facts (ii) and (iii), we may assume that L_i/K_i (i = 1, 2) is **tamely ramified**. Moreover, α induces the following commutative diagram (cf. Fact (iv)):

$$\begin{aligned} (\mathfrak{X}_1)_{k_{L_1}} &\curvearrowleft & G_1 \left(= \operatorname{Gal}(L_1/K_1) \right) \\ & \downarrow & & \downarrow \\ (\mathfrak{X}_2)_{k_{L_2}} &\curvearrowleft & G_2 \left(= \operatorname{Gal}(L_2/K_2) \right) \end{aligned}$$





When L_i/K_i is tamely ramified, we may calculate $i_{K_i}(\rho_i'^{-1}(P_i))$: P_i : a smooth point $\Longrightarrow i_{K_i}(\rho_i'^{-1}(P_i)) \equiv 1 \mod (q-1)$. P_i : a node and $p \neq 2 \Longrightarrow i_{K_i}(\rho_i'^{-1}(P_i)) \equiv 0 \text{ or } 2 \mod (q-1)$.

When L_i/K_i is tamely ramified, we may calculate $i_{K_i}({\rho'_i}^{-1}(P_i))$: P_i : a smooth point $\Longrightarrow i_{K_i}({\rho'_i}^{-1}(P_i)) \equiv 1 \mod (q-1)$. P_i : a node and $p \neq 2 \Longrightarrow i_{K_i}({\rho'_i}^{-1}(P_i)) \equiv 0 \text{ or } 2 \mod (q-1)$. $\Longrightarrow \text{ If } p \neq 2$, $i_{K_i}(X_i(K_i)) \equiv \sharp(\mathfrak{X}_i)_{K_{L_i}}^{\text{sm}}(k_{L_i})^{G_i} \mod 2$.

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Theorem B follows from Fact (iv).

Calculation of $i_{K_i}({\rho'_i}^{-1}(P_i))$

For simplicity, we omit subscripts i and assume that P is a smooth point.

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We have $\hat{\mathcal{O}}_{\mathfrak{X},P} \simeq \mathcal{O}_L[[T]]$ and

$$\rho^{-1}(P) \simeq \operatorname{Hom}_{\operatorname{Spec} \mathcal{O}_L}(\operatorname{Spec} \mathcal{O}_L, \operatorname{Spec} \mathcal{O}_{\mathfrak{X}, P})$$
$$\simeq \operatorname{Hom}_{\mathcal{O}_L}(\mathcal{O}_{\mathfrak{X}, P}, \mathcal{O}_L)$$
$$\simeq \operatorname{Hom}_{\mathcal{O}_L}(\hat{\mathcal{O}}_{\mathfrak{X}, P}, \mathcal{O}_L) \simeq \mathfrak{M}_L.$$
$$(f: \hat{\mathcal{O}}_{\mathfrak{X}, P} \simeq \mathcal{O}_L[[T]] \to \mathcal{O}_L) \mapsto f(T)$$

Fact (M)

(v) For $\gamma \in G$, the action $G \curvearrowright \hat{\mathcal{O}}_{\mathfrak{X},P} \simeq \mathcal{O}_L[[T]]$ is given by:

$$a \in \mathcal{O}_L \subset \mathcal{O}_L[[T]] \Longrightarrow \gamma \cdot a = \gamma(a) \text{ (usual Galois action)}$$
$$T \in \mathcal{O}_L[[T]] \Longrightarrow \gamma \cdot T = \sum_{i=0}^{\infty} b_i T^i \left(b_i \in \mathcal{O}_L, \, b_0 \in \mathfrak{M}_L, \, b_1 \in \mathcal{O}_L^{\times} \right)$$

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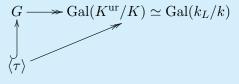
(vi) For $\gamma \in G$, the action $G \curvearrowright \rho^{-1}(P) \simeq \mathfrak{M}_L$ is given by:

$$[\gamma](x) := \gamma \cdot x = \sum_{i=0}^{\infty} \gamma(a_i x^i),$$

where
$$x \in \mathfrak{M}_L$$
 and $\gamma^{-1} \cdot T = \sum_{i=0}^{\infty} a_i T^i$.

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 \begin{split} \pi_K &: \text{ a uniformizer of } K \\ K^{\mathrm{ur}} &: \text{ the maximal unramified extension of } K \text{ in } L \\ \sigma \in G &: \text{ a generator of } \mathrm{Gal}(L/K^{\mathrm{ur}}) \\ e &:= [L:K^{\mathrm{ur}}] \end{split}
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$$\begin{split} \pi_K &: \text{ a uniformizer of } K \\ K^{\mathrm{ur}} &: \text{ the maximal unramified extension of } K \text{ in } L \\ \sigma &\in G &: \text{ a generator of } \mathrm{Gal}(L/K^{\mathrm{ur}}) \\ e &:= [L:K^{\mathrm{ur}}] \\ \end{split}$$
 Moreover, we take $\tau \in G$ such that:



Then,
$$G = \langle \sigma, \tau \rangle$$
.

The following theorem shows that we may take an isomorphism $\hat{\mathcal{O}}_{\mathfrak{X},P} \simeq \mathcal{O}_L[[T]]$ such that the action of G is not so complicated:

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Theorem (M)

 $\exists T\in \hat{\mathcal{O}}_{\mathfrak{X},P} \text{ such that } \hat{\mathcal{O}}_{\mathfrak{X},P}\simeq \mathcal{O}_L[[T]] \text{ and }$

$$\begin{cases} \sigma^{-1} \cdot T &= \omega T, \\ \tau^{-1} \cdot T &= \frac{1}{u} T. \end{cases}$$

Here, $\omega \in \mathcal{O}_L^{\times}$ is an *e*-th root of unity and $u \in \mathcal{O}_L^{\times}$. Moreover, $\exists x_0 \in \mathfrak{M}_L$ such that

$$\begin{cases} [\sigma](x_0) &= x_0, \\ [\tau](x_0) &= x_0. \end{cases}$$

The theorem, together with easy calculation, shows that:

The theorem, together with easy calculation, shows that:

 ${\rho'}^{-1}(P) = \rho^{-1}(P)^G \simeq \mathcal{O}_K x_0.$

In particular,

Corollary

$$i_K({\rho'}^{-1}(P)) \equiv 1 \mod (q-1).$$